## 5.1 Summary of EM theory

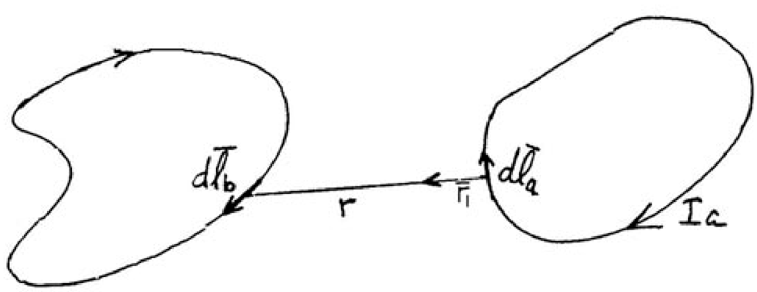
- Circuit theory
- Mutual and self inductance
- Sinusoidal oscillations, phase-shift and complex number representation
- The response of a loop target

Ampere formulated the idea of the force field when he found that there was a force created between two current carrying wires:

$$\text{Force / unit length} = k\frac{I_a I_b}{r}$$

In SI units, $k = \mu_0 / 2\pi$ where $\mu_0 = 4\pi\ 10^{-7}$

Later this was generalized so the force could be calculated between two current carrying circuits **a** and **b**.




$$F_{ab} = \frac{\mu_0}{4\pi} I_a I_b \oint\!\!\oint_{ab} \frac{d\vec{l}_b \times \left(d\vec{l}_a \times \vec{r}_1\right)}{r^2}$$

If we think of this force as something happening to circuit b caused by the total **effect** of circuit **a**, we could rewrite the expression for $\mathbf{F_{ab}}$ as

$$F_{ab} = I_b \oint_b d\vec{l}_b \times \left[ \frac{\mu_0}{4\pi} I_a \oint_a \frac{\left(d\vec{l}_a \times \vec{r}_1\right)}{r^2} \right]$$

The term in square brackets is now a property of **a**; it is something created by circuit **a** which interacts with the current in circuit **b** to produce the force on **b**. It is defined as a "field" and called the magnetic induction B.

In SI units, B is Tesla (T). In egs units, B is in Gauss (1.0 T = $10^4$ gauss). [In many magnetic surveys a cgs unit called the gamma ($\gamma$) is used. 1.0 $\gamma = 10^{-5}$ gauss = $10^{-9}$T (1.0 nT)]

The force on an element of current then has the form:

$$d\vec{F}_b = I_b d\vec{l}_b \times \vec{B}$$

and if there is a current density, J, i.e. a volume distribution of current , the expression for B becomes

$$\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \vec{r}_1}{r^2} dV$$

This simple formula for B was derived by Biot and Savart. The direction of the field follows the right hand rule – with the thumb of right hand in direction of current then fingers point in direction of B.

It may be shown that if one takes the curl of B from the formula, then:

$$\text{curl } B = \nabla \times B = \mu_0 J$$

which is called the differential form of Ampere's Law.

Faraday observed that a time varying magnetic field passing through a circuit produced an electromotive force, emf, that was proportional to the time rate of change of the magnetic flux threading the circuit. The total flux, $\Phi$, through the circuit is defined as the integral of the component of B normal to the surface contained by the circuit:

$$\Phi = \int \vec{B} \cdot d\vec{s}$$

Faraday's Law then states that an emf is produced in the circuit according to:

$$\text{emf} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s} = -\frac{\partial \Phi}{\partial t}$$

In a circuit the emf is the integral of the electric field around the circuit, i.e.

$$\text{emf} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

Using Stokes' Theorem we find the differential form of Faraday's Law:

$$\nabla \times E = -\frac{dB}{dt}.$$

With Ampere's Law, Faraday's Law, and the constitutive relations, $J = \sigma E$, $B = \mu H$, and the fact that $\nabla \cdot B = 0$, we have all the equations needed to solve any problem in low frequency electromagnetic induction. We begin with a loop circuit used to illustrate the fundamental physics of all the EM methods used to detect conductors in the ground. We first need to define an important circuit parameter - the inductance.

## Mutual and self inductance

We saw in the definition of magnetic field that a current in circuit 1 produces a magnetic field in the vicinity of circuit 2. The total flux, $\Phi$, through circuit 2 is defined as the integral of the component of B normal to the surface contained by the circuit:

$$\Phi_2 = \int_{S2} \vec{B} \cdot d\vec{s}$$

Faraday's Law then states that an emf is produced in circuit 2 if the flux threading it is changing with time, i.e.

$$\mathrm{emf}_2 = -\frac{\partial}{\partial t}\int_{S2} \vec{B} \cdot d\vec{s} = -\frac{\partial \Phi}{\partial t}.$$

The B threading circuit 2 is proportional to the current in circuit 1 so we can break $\frac{\partial \Phi}{\partial t}$ into $\frac{d\Phi_2}{dI_1} \cdot \frac{dI_1}{dt}$, and for any given circuit $\frac{d\Phi_2}{dI_1}$ is a constant and it is called the mutual inductance, $M_{12}$, of circuits 1 and 2.

A single isolated circuit carrying current also produces a magnetic field, which threads the circuit itself, so there is a relationship between the flux in the circuit and its own current. This is called the self inductance, L:

$$L = \frac{\partial \Phi_1}{\partial I_1}.$$

In any circuit the emf (voltage) caused by the changing field created by the current in the circuit is given by:

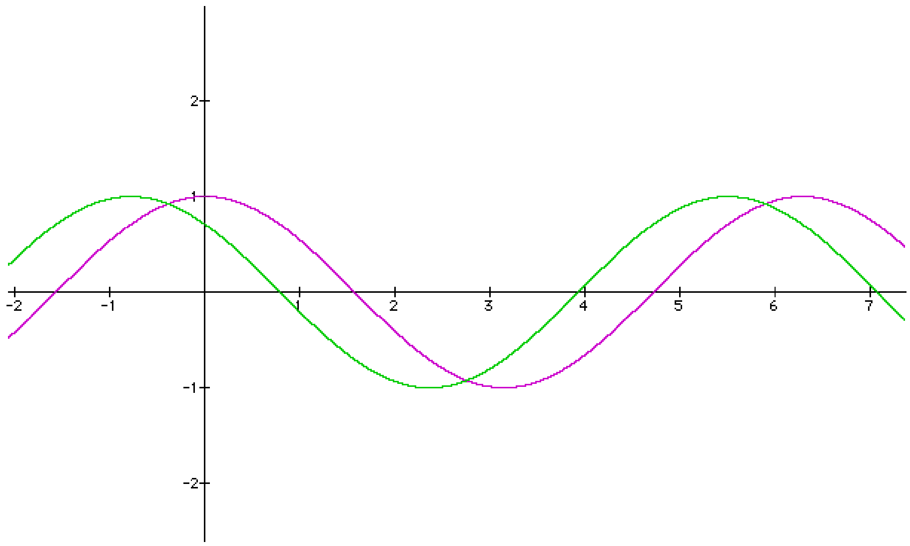
$$\text{emf} = V = L\frac{dI}{dt}.$$

This is the voltage created across an **inductor** when a time varying current flows through it.

We now face the fact that alternating currents are apparently going to produce voltages across certain circuit elements, which are proportional to the time derivative of the current. We need a fast review of the nomenclature of alternating fields, the concept of phase and the use of complex numbers to represent the fields.

## Sinusoidal oscillations, phase-shift and complex number representation

The two sinusoidal functions $\cos(\omega t)$ and $\cos(\omega t + \pi/4)$ are plotted below. The angular frequency $\omega$, in radians per second, is equal to $2\pi f$ when f is in cycles per second or Hertz (Hz). The period, T, is the time between two successive points of equal value on the curve, say between two successive maxima, and is equal to $1/f$ or $2\pi/\omega$. The $\cos(\omega t)$ curve, often called the reference curve, thus has its first maxima at 6.28 on the plot below. The second curve is said to have a phase shift with respect to the reference curve. In this case the phase shift is + π/4 radians or + 45°. It is observed that the curve with the positive phase shift peaks before the reference curve and it is said that it has a phase lead.




It is customary to use a complex number representation for operations with sinusoidal functions because the complexity of dealing with sums and products of the trigonometric functions is simplified with more easily manipulated functions.

The basic identity used is Euler's formula:

$$e^{i\phi} = \cos\phi + i \sin\phi \quad \text{where } i = \sqrt{-1}$$

We may then write $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$ and at the end of all operations we simply take the real part of the result. Re $e^{i\omega t}$ is $\cos(\omega t)$. We immediately can express the phase shift in the same complex notation via:

$$\cos(\omega t + \phi) = \mathrm{Re}[\, e^{i(\omega t + \phi)}] = \mathrm{Re}[e^{i\omega t}\, e^{i\phi}\,]$$

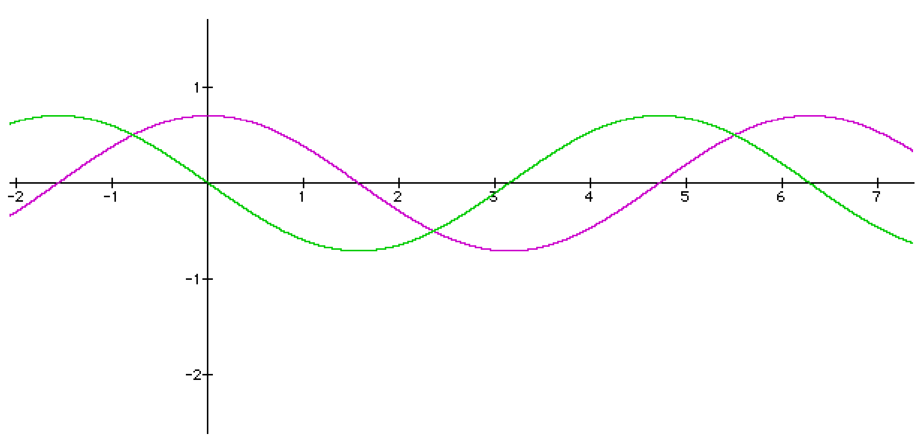
$$= \mathrm{Re}[(\cos(\omega t) + i \sin(\omega t))\cdot(\cos \phi + i \sin \phi)]$$

$$= \cos\phi \cos\omega t - \sin\phi \sin\omega t$$

$$= a \cos\omega t - b \sin\omega t$$

So we have found with this example of complex number representation that the phase shifted cosine may be equally well represented by the sum of a cosine wave and a sine wave whose amplitudes are the cosine and sine of the phase shift respectively. In this representation a is the amplitude of the real or **in-phase** component of the signal and b is the amplitude of the imaginary or **out-of-phase** (also called the **quadrature**) component of the signal. Many signal detection systems are capable of measuring the real and quadrature components of an arbitrary waveform. The phase shift is then derived from: $\phi = \tan^{-1}(b/a)$ and the amplitude is $(a^2 + b^2)^{1/2} = 1.0$.

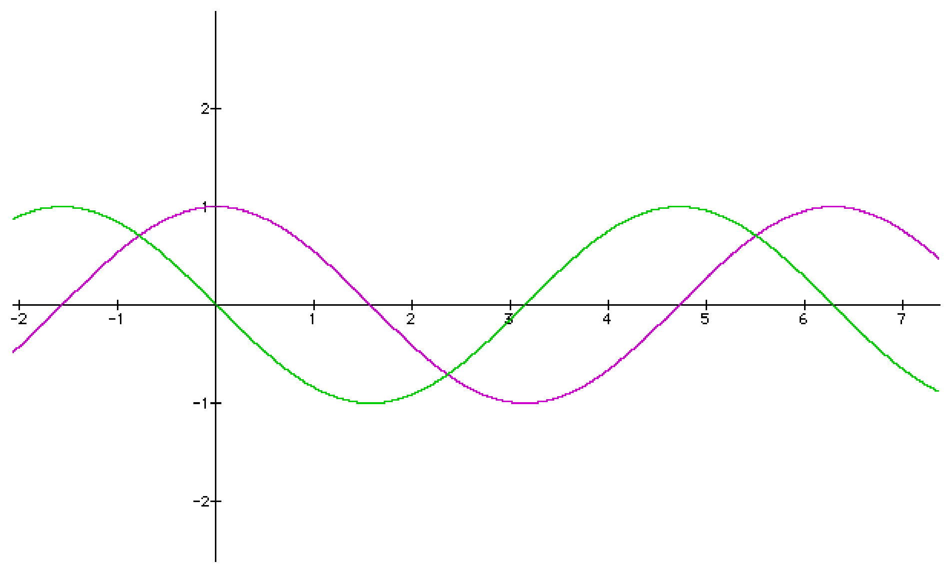
For the phase shift of 45° in the above plot, the amplitudes of the in-phase and quadrature components would be $1/\sqrt{2}$, or 0.707. These two waveforms are plotted below. Added together they would equal the phase shifted waveform in the first graph.



Finally, derivatives are easily done in complex notation via:

$$\frac{\partial}{\partial t}\left(e^{i\omega t}\right)=i\omega e^{i\omega t}=i\omega\left(\cos\omega t+i\sin\omega t\right)=-\omega\sin\omega t+i\omega\cos\omega t$$

The derivative of cosine is minus the sine multiplied by the frequency. The derivative is plotted below.




In the context of inductance, we find that the voltage across in inductor is phase shifted by + 90°, and multiplied by $\omega$.

With these basics, we can now analyze the famous shorted turn or loop.

## The response of a loop target

A loop of wire, also called a shorted turn, is a simple target for illustrating the electromagnetic response of a conducting object in the ground. All active EM systems include an alternating current source, the transmitter (T), which produces a primary alternating magnetic field $B_0$. This alternating magnetic field induces currents in any nearby conductor by virtue of Faraday's law. These induced currents in turn produce secondary magnetic fields, $B^{sec}$, which are measured by the receiver, R. The current source and the receivers are usually multiturn loops of wire. The field produced from a small multiturn loop is proportional to its dipole moment, M, which is equal the product of the current, I, the area of the loop, A, and the number of turns, N. The moment is a vector whose direction is normal to the plane of the loop (along the axis of the loop). The currents induced in an object in the ground are a function of the time rate of change of the primary field at the object and of its size, shape, conductivity ($\sigma$) and magnetic permeability ($\mu$), and the conductivity and permeability of the surrounding ground. The **response** of a particular object is defined as the measured field for a given configuration of transmitter and receiver.

In the following sketch a **transmitter,** $T_x$ (of moment $M_x$ where the moment is the product of the area, number of turns and the current) carries an alternating current $I_0 e^{i\omega t}$ (The subscript indicates the direction of the axis of the loop). This transmitter produces a magnetic dipole field given by:

$$\vec{B} = \frac{\mu_0}{4\pi} \nabla\left(\vec{m} \cdot \nabla\left(\frac{1}{r}\right)\right)$$

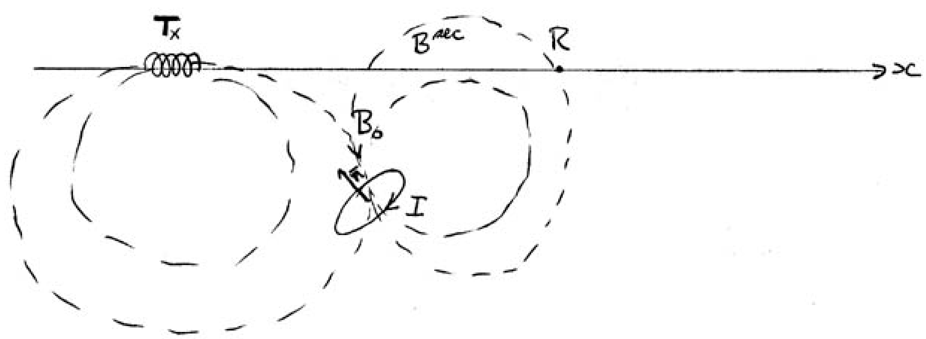
or expanded in spherical coordinates,

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2(\pi a^2 I)}{r^3}\cos\theta\,\vec{r} + \frac{\mu_0}{4\pi}\frac{2(\pi a^2 I)}{r^3}\sin\theta\,\vec{\theta}$$

For the x-directed transmitter in the following sketch the dipole field can be expanded in rectangular coordinates via:

$$\vec{B} = \mu_0 M_x \frac{\left(2x^2 - y^2 - z^2\right)\vec{u}_x + 3xy\,\vec{u}_y + 3xz\,u_z}{r^5}$$

where $\vec{u}_x, \vec{u}_y$ and $\vec{u}_z$ are unit vectors in the x , y and z directions respectively.




The primary field from the transmitter passes through the target loop. Assuming the loop radius is small the total flux normal to the loop, Φ, is ,

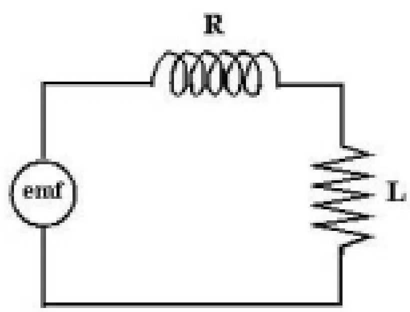
$$\Phi = B_0 e^{i\omega t} A \cdot \vec{n}$$

where A is the area of the loop and $\bar{n}$ is a unit normal to the plane of the target loop.

The time rate of change of the flux normal to the loop produces an emf in the loop by Faraday's Law:

$$\text{emf} = -\frac{\partial \Phi}{\partial t}$$

This emf drives a current through the resistance and inductance of the loop as shown in the equivalent circuit shown below.




The sum of the emf and the voltage drops across the resistance and inductance must be zero, so we have an equation for the current in the loop:

$$-\frac{\partial \Phi}{\partial t} + IR + L\frac{\partial I}{\partial t} = 0$$

In the frequency domain with an $e^{iwt}$ time dependence this equation becomes:

$$-i\omega B_0 A + IR + i\omega LI = 0$$

and so $$I = \frac{i\omega B_0 A}{R + i\omega L} = B_0 A\left(\frac{i\omega R}{R^2 + \omega^2 L^2} + \frac{\omega^2 L}{R^2 + \omega^2 L^2}\right)$$

The ratio of L to R is called the time constant, $\tau$, of the LR circuit and with this definition the current becomes:

$$I = \frac{B_0 A}{L}\left(\frac{i\omega\tau}{1+(\omega\tau)^2} + \frac{(\omega\tau)^2}{1+(\omega\tau)^2}\right) = \frac{\Phi}{L}(M + iN)$$

The current induced in the loop has a component in-phase with the inducing field and a component 90°(in quadrature) to it. The relative amount of these real and imaginary components may be seen by plotting M and N as a function of frequency as seen in Figure 5.1a, b, c. The three plots illustrate the common methods of plotting EM responses.

The response of this simple loop target has several important characteristics:

i) at low values $\omega\tau$ the quadrature response is linear in $\omega\tau$, and the real response is proportional to $(\omega\tau)^2$. The quadrature response is larger than the real at low frequencies because the $L\frac{\partial I}{\partial t}$ term is small compared to the IR term in $IR + L\frac{\partial I}{\partial t}$ and so the current is simply proportional to $i\omega\Phi/R$. This is called the **resistive limit**.

ii) at high $\omega\tau$ the real current asymptotes to $\Phi/L$, and the quadrature component is zero. This is called the **inductive limit**. The high frequency current is independent of the resistance of the loop and depends only on its size and geometry. The sense of the current in the loop is such as to keep the flux from passing through the loop.

Figure 5.1c is an alternate means of showing the real and quadrature components M and N as a function of $\omega\tau$ as a single curve (this is called an Argand or Bode plot).

The transient response of the loop when a steady, DC, field is abruptly turned off (called the step-function response) is given by:

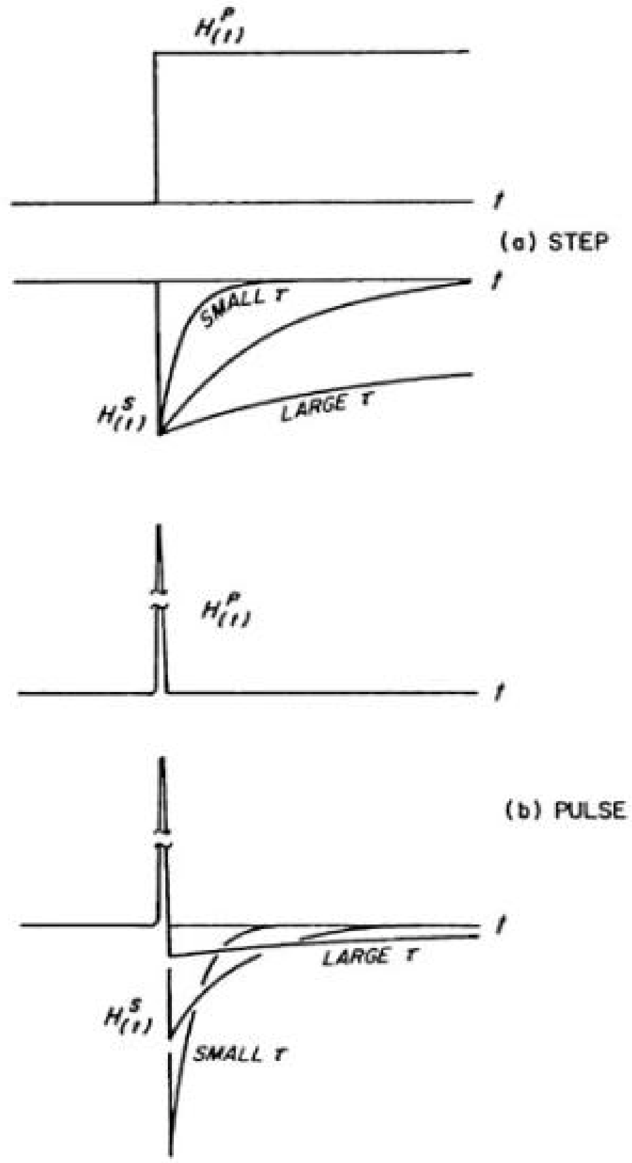
$$I(t) = \frac{B_0 A}{L} u(t) e^{-\frac{t}{\tau}} = \frac{\Phi}{L} u(t) e^{-\frac{t}{\tau}} \qquad u(t) = 0,\ t<0:\ u(t)=1,\ t\geq 0$$

The impulse response is given by:

$$I(t) = \frac{\Phi}{L}\left(\delta(t) - \frac{e^{-t/\tau}}{\tau}\right)$$

The current in the single turn loop creates a dipole moment IA which in turn produces a secondary magnetic field, $B^{sec}$, which is detected by a suitable **receiver,** R.

The secondary field from a loop excited by an impulse function is the same as the time rate of change of field at the receiver due to a step function. Since most EM receivers measure the time rate of change of field the impulse response is commonly the response that is measured.




Transient current induced in a simple circuit by (a) a step or (b) an impulse in primary magnetic field. (From West and Macnae, 1991)